# Matrix realignment and partial transpose approach to entangling power of quantum evolutions

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Based on the matrix realignment and partial transpose, we develop an approach to entangling power and operator entanglement of quantum unitary operators. We demonstrate efficiency of the approach by studying several unitary operators on qudits, and indicate that these two matrix rearrangements are not only powerful for studying separability problem of quantum states, but also useful in studying entangling capabilities of quantum operators.

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Given a unitary operator, in the context of quantum information [1], one may ask how much entanglement capability does the operator have. The entangling unitary operator can be considered as a resource for quantum information processing, and it becomes important to quantitatively describe unitary operators. Recently, there are increasing interests in the entanglement capabilities of quantum evolutions and Hamiltonians [2]-[10]. The entangling power based on the linear entropy [2] is a valuable, and relatively easy to calculate, measure of the entanglement capability of an operator. The entangling power for two qudits can be expressed in terms of operator entanglement [3, 7] (also called Schmidt strength [11]). Both entangling power and operator entanglement have been applied to the study of quantum chaotic systems [12, 13, 14, 15]. Moreover, the concept of entangling power has been extended to the case with ancillas [16], the case of entanglement-changing power [17], and the case of disentangling power [18].

Let us start by introducing some basics of entanglement of quantum states, the operator entanglement, and the entangling power. For a two-qudit pure state $|\Psi\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$, one can quantify entanglement by using the linear entropy

$$E(|\Psi\rangle) := 1 - \mathrm{Tr}\rho_1^2, \tag{1}$$

where $\rho_1 = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|)$ is the reduced density matrix. The linear entropy satisfy the inequalities $0 \leq E(|\Psi\rangle) \leq 1 - 1/d$, where the lower (upper) bound is reached if and only if $|\Psi\rangle$ is a product state (maximally entangled state).

In the orthogonal basis $\{|1\rangle, ..., |d\rangle\}$, state $|\Psi\rangle$ is written as

$$|\Psi\rangle = \sum_{i,j=1}^{d} A_{ij}|i\rangle \otimes |j\rangle, \tag{2}$$

where $A_{ij}$ are the coefficients, and $A$ can be considered as a matrix. After direct calcualtions, one find that the reduced density matrix $\rho_1 = AA^\dagger$. Substituting it to Eq. (1) leads to another expression of the linear entropy

$$E(|\Psi\rangle) = 1 - \mathrm{Tr}\left(AA^\dagger AA^\dagger\right), \tag{3}$$

which will be used for later discussions.

An operator can increase entanglement of a state, but an operator can also be considered to be entangled because operators themselves inhabit a Hilbert space. The entanglement of quantum operators is introduced [3] by noting that the linear operators over $\mathcal{H}_d$ span a $d^2$-dimensional Hilbert space with the scalar product between two operators $X$ and $Y$ given by the Hilbert-Schmidt product $\langle X, Y\rangle := \mathrm{Tr}(X^\dagger Y)$, and $||X||_{\mathrm{HS}} := \sqrt{\mathrm{Tr}(X^\dagger X)}$. We denote this $d^2$-dimensional Hilbert space as $\mathcal{H}_{d^2}^{\mathrm{HS}}$. Thus, the operator acting on $\mathcal{H}_d \otimes \mathcal{H}_d$ is a state in the composite Hilbert space $\mathcal{H}_{d^2}^{\mathrm{HS}} \otimes \mathcal{H}_{d^2}^{\mathrm{HS}}$, and the entanglement of an operator $X$ is well-defined [3].

The entangling power quantifies the entanglement capability of a unitary operator $U$. It is defined as [2]

$$e_{\mathrm{p}}(U) := \overline{E(U|\psi_1\rangle \otimes |\psi_2\rangle)}. \tag{4}$$

It tells us how much entanglement the operator produces, on average, when acting on product states. After a suitable average over initial product states, one find [2]

$$e_{\mathrm{p}}(U) = \left(\frac{d}{d+1}\right)^2 [E(U) + E(US_{12}) - E(S_{12})]. \tag{5}$$

Thus, the entangling power defined on $d \times d$ systems can be expressed in terms of the entanglement of three operators, $U$, $US_{12}$, and $S_{12}$. Here, $S_{12}$ is the swappig operator. Therefore, by studying the entanglement of these three operators we can obtain the entangling power of $U$.

Next, we give our approach, and first consider the operator entanglement of a unitary operator. A unitary operator can be written as

$$U = \sum_{ijkl} \langle ij|U|kl\rangle |ij\rangle\langle kl| \tag{6}$$

$$= \sum_{ijkl} U_{ij,kl}|i\rangle\langle k| \otimes |j\rangle\langle l| \tag{7}$$

$$= \sum_{ijkl} U_{ij,kl} e_{ik} \otimes e_{jl}, \tag{8}$$

where $e_{ik}$ are orthogonal basis in the space $\mathcal{H}_{d^2}^{\text{HS}}$, and they can be considered as states. Now, we define a new matrix $U^R$ as

$$\left(U^R\right)_{ij,kl} = U_{ik,jl}. \qquad (9)$$

The matrix can be obtained by realigment of matrix $U$. Using this realigned matrix, one can express normalized unitary operator $\tilde{U}$ as

$$\tilde{U} = \frac{U}{d} = \frac{1}{d}\sum_{ijkl}\left(U^R\right)_{ik,jl} e_{ik}\otimes e_{jl}, \qquad (10)$$

Comparing Eqs.(2) and (10), and using Eq. (3), one obtain the operator entanglement of $U$

$$E(U) = 1 - \frac{1}{d^4}\text{Tr}(U^R\left(U^R\right)^{\dagger}U^R\left(U^R\right)^{\dagger}). \qquad (11)$$

We see that the operator entanglement is determined by the naturally appeared realigned matrix. The realigned matrix is easy to obtain from the original unitary matrix, and thus our approach is very efficient to study operator entanglement.

It is more interesting to see that this matrix realigment is the same as density matrix realigment when studying the separability problem of quantum mixed state [19]. The realignment criteria (also called cross norm criteria) is very strong to detect many bound entangled states. We see here that the same matrix realignment approch is very effective in studying operator entanglement.

There is another matrix rearrangement, called partial transpose [20]. A partial transpose with respect to the first system $U^{T_1}$ is defined as

$$\left(U^{T_1}\right)_{ij,kl} = U_{kj,il}. \qquad (12)$$

It is well-known that the partial transposed method is very useful in studying entanglement of quantum mixed states. Was it useful in studying operator entangling properties? We will see that indeed it is.

The entangling power is determined by three operator entanglement $E_l(U), E_l(S_{12})$, and $E_l(S_{12}U)$. The first two can be determined by the realignment method, and the last one is of course can be determined by the same method, but with extra effort to make matrix multiplication $S_{12}U$. In fact, we have [21]

$$S_{12}\left(S_{12}U\right)^R = U^{T_1}.$$

Using the above property and applying Eq.(11) to $S_{12}U$, we obtain

$$E(S_{12}U) = 1 - \frac{1}{d^4}\text{Tr}(U^{T_1}\left(U^{T_1}\right)^{\dagger}U^{T_1}\left(U^{T_1}\right)^{\dagger}) \qquad (13)$$

Therefore, the operator entanglement of $S_{12}U$ can be written in terms of partial transposed unitary matrix $U^{T_1}$.

From Eqs. (5), (11), and (13), we know that the entangling power can be determined by the matrix realignment and the partial transpose

$$e_{\text{p}}(U) = \left(\frac{d}{d+1}\right)^2[2 - E(S_{12})]$$
$$- \frac{1}{(d+1)^2d^2}\text{Tr}(\left[U^R\left(U^R\right)^{\dagger}\right]^2 + \left[U^{T_1}\left(U^{T_1}\right)^{\dagger}\right]^2). \qquad (14)$$

Both these matrix manipulations are powerful in the context of separability of quantum states. Here, we find they are also powerful in studying operator entanglement and entangling power in quantum information theory.

To illustrate the efficiency of the approach, let us consider several examples.

*Example 1*: The swap operator $S_{12}$. It can be written as

$$S_{12} = \sum_{i,j=1}^{d}|ij\rangle\langle ji|. \qquad (15)$$

It is easy to see that

$$S_{12}^R = S_{12}, S_{12}^{\dagger} = S_{12}, S_{12}^2 = I.$$

The swap operator is invariant under the matrix realigment. Then, from Eq.(11), the linear entropy of the swap operator is given by

$$E_l(S_{12}) = 1 - \frac{1}{d^4}\text{Tr}(S_{12}^4) = 1 - \frac{1}{d^2}. \qquad (16)$$

From Eq.(5), evidently the entangling power of the swap operator is zero.

*Example 2*: The unitary operator $V$ generated by the swap

$$V = \exp(-itS_{12}) = \cos(t)I - i\sin(t)S_{12}. \qquad (17)$$

It is straightfoward to check the following identities

$$I^{T_1} = I, I^R = dP_+, S_{12}^{T_1} = dP_+, S_{12}^R = S, \qquad (18)$$

where projector

$$P_+ = |\Psi_+\rangle\langle\Psi_+|, |\Psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle\otimes|i\rangle. \qquad (19)$$

From the above identies, we obtain

$$V^R(t) = \cos(t)dP_+ - i\sin(t)S_{12},$$
$$V^{T_1}(t) = \cos(t)I - i\sin(t)dP_+. \qquad (20)$$

Then, we find

$$V^R(t)\left[V^R(t)\right]^{\dagger} = \cos^2(t)d^2P_+ + \sin^2(t)I,$$
$$V^{T_1}(t)\left[V^{T_1}(t)\right]^{\dagger} = \cos^2(t)I + \sin^2(t)d^2P_+. \qquad (21)$$

From the above two equations and Eqs. (11), and (13), we find linear entropies

$$E(V) = \left(1 - \frac{1}{d^2}\right)(1 - \cos^4 t) \quad (22)$$

$$E(VS_{12}) = \left(1 - \frac{1}{d^2}\right)(1 - \sin^4 t) \quad (23)$$

Substuting Eqs.(22)and (23) into (5) leads to the expression of the entangling power

$$e_{\mathrm{p}} = \frac{d^2 - 1}{2(d+1)^2}\sin^2(2t). \quad (24)$$

From Eqs.(22) and (24), we see that the maximal value of the operator entanglement occurs at $t = \frac{\pi}{2}$, however, at this point the entangling is zero. This point corresonds to the swap. The maximal entangling power occurs at $t = \frac{\pi}{4}$, which corresponds to the $\sqrt{swap}$ gate, the square of which is just the swap gate. Thus, the $\sqrt{swap}$ gate can be used as an important gate for quantum computing not only in qubit systems, but also in qudit systems. Quantiatively, the operator entanglement and entangling power of the $\sqrt{swap}$ gate is given by

$$E(V) = \frac{3}{4}\left(1 - \frac{1}{d^2}\right), e_{\mathrm{p}} = \frac{d^2 - 1}{2 \cdot (d+1)^2}. \quad (25)$$

respectively.
*Example 3*: A general two-qudit controlled-$U$ gate is given by

$$C_U := \sum_{n=1}^{d} |n\rangle\langle n| \otimes U_n, \quad (26)$$

The controlled-$U$ gate implements the unitary operator $U_n$ on the second system if and only if the first system is in the state $|n\rangle$. For the controlled-$U$ operation, it was found that [16]

$$e_{\mathrm{p}}(C_U) = \left(\frac{d}{d+1}\right)^2 E(C_U), \quad (27)$$

Let us prove this via our approach. From Eq.(5), to prove the above identity is equivalent to prove that

$$E(C_U S_{12}) = E(S_{12}). \quad (28)$$

In fact, we have a more general result that *if the partial transpose of a unitary operator $U$ is still an unitary operator, then, $E(US_{12}) = 1 - 1/d^2 = E(S_{12})$.*This result immediately follows from Eq. (13). For our operator $C_U$, from the definition, it is not diffulcult to see that it is invariant under the partial transpose with respect to the first system. Of course, $C_U$ is unitary, and then Eq.(28) holds. In this case, the entangling power is proportional to the operator entanglement of the controlled-$U$ gate. It is easier to obtain Eq.(28) via our approach.

In conclusion, we have developed an efficient way for studying entangling power and operator entanglement. One only needs to obtain the realigned unitary operator and partially transposed operator to determine the entangling power. Once we have analytical expression for a unitary matrix, then analytical expressions for entangling power and operator entanglement can be obtained. If we cannot have the analytical expression, it is very convenient to make the matrix rearrangements numerically, and then entangling power and operator entanglement can be quickly computed.

The matrix realignment and partial transpose play very important roles in the theory of separability of quantum mixed states, and we see here that they naturally appears in the study of entanglement capabilities of quantum evolutions. The approach developed here can be applied to investigate entanglement capabilities in many physical systems such as quantum chaotic systems.

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